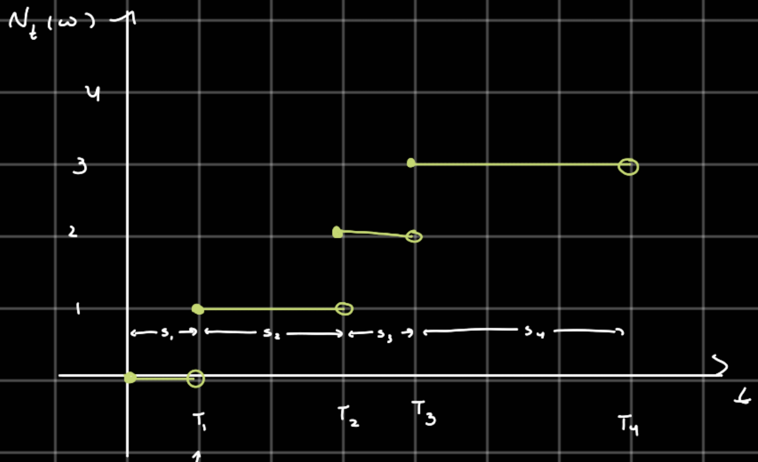
14: Poisson Processes

- good at capturing idea of random arrivals
  - $\rightsquigarrow$ eg: Geiger counter $\rightarrow$ fcn of continuous time; exactly a Poisson process
  - or toll booth, bus arrivals

Construction of Pois Proc

Step 1: Counting Process



T_i's are arrival times

S_i's are interarrival times

Counting Process Properties:
- continuous from the right; count the arrival when in happens
- takes integer values
- starts at 0

- Poisson process with rate λ ($PP(\lambda)$) is a counting process with iid $Exp(\lambda)$ interarrival times

Q/ what's the connection btwn Poisson rv's & and Poisson processes?

A/

Theorem: If $(N_t)_{t \geq 0}$ is a $PP(\lambda)$ then

$$N_t \sim Poisson(\lambda t) \quad \forall t \geq 0$$

"the # of arrivals are Poisson"

$$P\{N_t = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n \geq 0$$

Pf:

$P\{N_t = n\} = P\{$ n^{th} arrival arrived before t & $(n+1)^{th}$ arrival hasn't yet appeared$\} = P\{T_n \leq t \leq T_{n+1}\}$

$$= \mathbb{E}[\mathbb{1}_{\{T_n \leq t\}} \mathbb{1}_{\{t \leq T_{n+1}\}}$$

$$= \int f_{T_n}(s)\, \mathbb{1}_{\{T_n \le t\}}\, \mathbb{E}[\mathbb{1}_{\{t < T_{n+1}\}} \mid T_n = s\}\, ds$$

$$= \int f_{T_n}(s)\, \mathbb{1}_{\{T_n \le t\}}\, \mathbb{E}[\mathbb{1}_{\{t < T_n + S_{n+1}\}} \mid T_n = s\}\, ds$$

$$= \int f_{T_n}(s)\, \mathbb{1}_{\{s \le t\}}\, \mathbb{E}[\mathbb{1}_{\{t < s + S_{n+1}\}} \mid T_n = s\}\, ds$$

$$= \int f_{T_n}(s)\, \mathbb{1}_{\{s \le t\}}\, \underbrace{\mathbb{E}[\mathbb{1}_{\{t < s + S_{n+1}\}}]}_{e^{-\lambda(t-s)}}\, ds$$

$$= \int_0^t f_{T_n}(s)\, \underbrace{\mathbb{1}_{\{T_n \le t\}}}_{=1 \text{ based on bounds of integrals}}\, e^{-\lambda(t-s)}\, ds$$

$$= \int_0^t f_{T_n}(s)\, e^{-\lambda(t-s)}\, ds$$

$$T_n \sim \text{Erlang} \longrightarrow f_{T_n}(s) = \frac{\lambda e^{-\lambda s}(\lambda s)^{n-1}}{(n-1)!}$$

$$= \int_0^t \frac{\lambda e^{-\lambda s}(\lambda s)^{n-1}}{(n-1)!}\, e^{-\lambda(t-s)}\, ds$$

$$= \frac{\lambda^n e^{-\lambda}}{(n-1)!} \int_0^t s^{n-1}\, ds$$

$$= \underbrace{\frac{(\lambda t)^n e^{-\lambda t}}{n!}}_{\text{exactly pmf for Poisson distribution}} \qquad n \geq 0$$

- If $(N_t)_{t\geq 0} \sim PP(\lambda)$, then for any $s \geq 0$

  $$\underbrace{(N_{t+s} - N_s)_{t\geq 0}}_{\text{essentially re-indexing}} \sim PP(\lambda)$$

  ↳ start at time s, the new PP starting at that time is still $PP(\lambda)$ due to memoryless prop

$\Rightarrow$ increments of $PP(\lambda)$ are stationary

$$(N_{t+s} - N_s) \overset{d}{=} (N_{t+\tau} - N_\tau) \qquad \forall t, \tau, s \geq 0$$

- Poisson processes have independent increments

ie, if $t_0 < t_1 < \cdots < t_k$

$\Rightarrow$ increments $(N_{t_1} - N_{t_0}),\ (N_{t_2} - N_{t_1}),\ \cdots\ (N_{t_k} - N_{t_{k-1}})$ are indep.

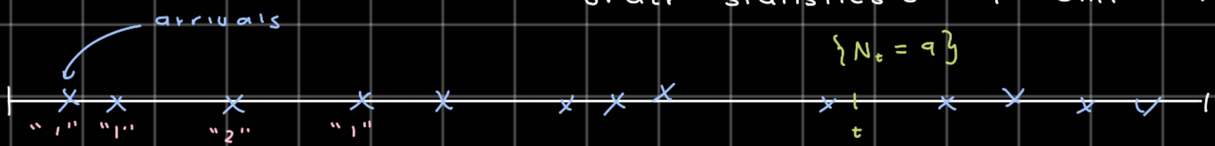
Thm: If $(N_t)_{t\geq 0}$ is a counting process with independent, stationary increments and $N_t \sim \text{Poisson}(\lambda t)$, then $(N_t)_{t\geq 0} \sim PP(\lambda)$

Conditional Distribution of Arrivals

Thm: Conditioned on $\{N_t = n\}$, the arrival times of these n arrivals is equal in distribution to the order statistics of iid Uniform rvs:

$$(T_1, T_2, \ldots, T_n) \overset{d}{=} \underbrace{(U_{(1)}, \ldots, U_{(n)})}_{\text{order statistics of } U_i \sim \text{Unif}(0,t)}$$

Note: we're only using order statistics bc of the order of arrivals, eg 1^{st} arrives 1^{st}, ...



Pf: Fix $0 = t_0 \leq t_1 \leq \cdots \leq t_n$

$$f_{T_1 \cdots T_N | N_t}(t_1, \ldots, t_n | n) \overset{\text{cts Bayes}}{=} \frac{P\{N_t = n \mid T_1 = t_1, \ldots T_n = t_n\}}{P\{N_t = n\}} f_{T_1 \cdots T_N}(t_1, \ldots, t_n)$$

$$= \frac{P\{N_t - N_{t_n} = 0 \mid \cancel{T_1 = t_1 \cdots T_n = t_n}\}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \prod_{i=1}^{N} f_{S_i}(t_i - t_{i-1})$$

by independence

$$= \frac{e^{-\lambda(t - t_n)}}{\frac{e^{-\lambda t}(\lambda t)^n}{n!}} \prod_{i=1}^{n} \lambda e^{\underbrace{-\lambda(t_i - t_{i-1})}}$$

product $\Rightarrow$ Σ in the exponent
$\Rightarrow$ nice telescoping that lets us cancel

$$= \frac{n!}{t^n}$$

$n!$ → comes from the ordering of the arrivals

t^n → from the iids

Example: Let cars pass through tollbooth $\sim PP(\lambda)$

Q/ What is Prob that no cars pass in 2 mins?

A/ $e^{-2\lambda}$

↳ $P\{N_2 = 0\} = \frac{(\lambda 2)^0 e^{-2\lambda}}{0!} = e^{-2\lambda}$

Q/ If 10 vehicles pass in 2 mins whats distribution of vehicles that passed in 1st 30 secs?

A/ $\text{Bin}(10, \frac{1}{4})$

Merging and Splitting (Superposition and Thinning)

Merging: If $(N_{1,t})_{t\geq 0} \sim PP(\lambda) \underbrace{\perp\!\!\!\perp}_{\text{indep.}} (N_{2,t})_{t\geq 0} \sim PP(\mu)$

then $(N_{1,t} + N_{2,t})_{t\geq 0} \sim PP(\lambda + \mu)$

Pf: $N_{1,0} + N_{2,0} = 0$

- $(N_{1,t} + N_{2,t})_{t\geq 0}$ is a counting process w/ independent increments
- Check distribution of increments:

$$(N_{1,t+s} + N_{2,t+s}) - (N_{1,s} + N_{2,s}) = (N_{1,t+s} - N_{1,s}) + (N_{2,t+s} - N_{2,s})$$

$$\overset{d}{=} \text{Poisson}(\lambda t) \underbrace{*}_{\text{convolution}} \text{Poisson}(\lambda \mu) = \text{Poisson}((\lambda + \mu)t)$$

Splitting: Let $(N_t)_{t\geq 0} \sim PP(\lambda)$

Independently mark arrival "1" wp p.

" " " "2" " $(1-p)$

Let $(N_{i,t})_{t\geq 0}$ be process that counts arrivals marked w/ i ($i = 1, 2$)

Thm: $(N_{1,t})_{t\geq 0} \sim PP(p\lambda) \perp\!\!\!\perp (N_{2,t})_{t\geq 0} \sim PP((1-p)\lambda)$